Partial Differential Equations Final Exam

April 11, 2018

You have 3 hours to complete this exam. The exam is worth 100 points. Please show all work. Each question has a point value assigned next to it for a total of 90 points (10 points free). Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

1. (10 points) a) Find complex coordinates $(u(x,y), v(x,y))$ so that the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ becomes $\phi_{uv} = 0$.
   (5 points) b) Prove that a solution of Laplace's equation can be written $\phi(x,y) = F(u) + G(v)$ for some complex functions F, G, where u, v are the coordinates.
   (5 points) c) Express the solution $\phi(x,y) = xy$ in the form $F(u) + G(v)$.

2. Consider the partial differential equation for $\theta(x,t)$ with $0 \leq x \leq \pi$ and $t \geq 0$

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \qquad \frac{\partial \theta}{\partial x}(0,t) = 0 \qquad \theta(\pi, t) = 0$$

   (8 points) a) With a careful explanation of each step of your argument use the method of separation of variables to show that the solution has the form

$$\theta(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2 t} \cos\left(\left(n + \frac{1}{2}\right) x\right) \tag{0.1}$$

   (5 points) b) If $\theta(x,0) = 1$, then show that

$$A_n = \frac{2}{\pi} \frac{(-1)^n}{\left(n + \frac{1}{2}\right)}$$

   (5 points) c) By integrating (0.1) with respect to x under the condition $\theta(x,0) = 1$ from part b) show that

$$\frac{d}{dt} \int_0^{\pi} \theta(x,t)\, dx = \frac{\partial \theta}{\partial x}(\pi, t) = -\frac{2}{\pi} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})^2 t} \tag{0.2}$$

(2 points) d) Comment on the convergence of this sum for finite t.

3. A prisoner is attached to an infinitely long chain parametrised by a coordinate $x \in [0, \infty)$. At time $t = 0$ the prisoner starts jumping up and down on the spot $x = 0$ so that his height at time $t \geq 0$ is $y(0, t) = 1 - \cos(t)$. The chain starts off with

$$y(x, 0) = 0 \quad x \geq 0, \qquad \partial_t y(x, 0) = 0, \quad x \geq 0$$

and obeys the wave equation $y_{tt} = y_{xx}$. By substituting D'Alembert's solution $y(x, t) = F(x+t) + G(x-t)$ into the initial conditions show that:
a) (5 points) For some constant k, $F(z) = k$ and $G(z) = -k$ for all $z \geq 0$.
b) (5 points) Using the condition $y(0, t) = 1 - \cos t$ for $t \geq 0$ prove that $G(z) = 1 - k - \cos(z)$ for $z < 0$.
c) (5 points) Find $y(x, t)$ for all $x \geq 0$ and $t \geq 0$.
d) (5 points) A prison guard is sitting at $x = 8$ watching the chain. At what point does he notice the prisoner is jumping up and down?

4. (20 points total) Recall that if we have a bounded 2π periodic function

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

the N^{th} partial sum is

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

Prove that $S_N(x)$ goes to $f(x)$ pointwise as $N \to \infty$ if $f(x)$ has a continuous derivative. Hint: Show that

$\sin\left(\left(N+\frac{1}{2}\right)x\right)$

$$\sum_{n=-N}^{N} e^{inx} = \frac{\cancel{\sin \frac{(N+1)x}{2}}}{\sin \frac{x}{2}}$$

and use this to help you rewrite $S_N(x)$

5. Suppose that $u(r, \theta)$ is a harmonic function on the closed disk $D = \{r \leq 1\}$ and that

$$u(1, \theta) = 6\cos(2\theta) - 2$$

(5 points) a) What are the maximum and minimum values of u in D?
(5 points) b) Find the value of u at $r = 0$.

Some useful equations: General solution of the wave equation $u_{tt} = c^2 u_{xx}$

$$u(x,t) = f(x+ct) + g(x-ct)$$

Laplace operator in Cartesian coordinates

$$\Delta = \partial_x^2 + \partial_y^2$$

Poisson's formula

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \, d\phi$$

Fourier cosine series:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

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